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A spin foam model for pure gauge theory coupled to quantum gravity

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Abstract

We propose a spin foam model for pure gauge fields coupled to Riemannian quantum gravity in four dimensions. The model is formulated for the triangulation of a four-manifold which is given merely combinatorially. The Riemannian Barrett–Crane model provides the gravity sector of our model and dynamically assigns geometric data to the given combinatorial triangulation. The gauge theory sector is a lattice gauge theory living on the same triangulation and obtains from the gravity sector the geometric information which is required to calculate the Yang–Mills action. The model is designed so that one obtains a continuum approximation of the gauge theory sector at an effective level, similarly to the continuum limit of lattice gauge theory, when the typical length scale of gravity is much smaller than the Yang–Mills scale.

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1 Introduction

Spin foam models were introduced in a non-perturbative approach to quantum gravity, inspired by ideas and results from loop quantum gravity, topological quantum field theory and lattice gauge theory, as an attempt to define a manifestly background independent formulation of quantum gravity. A spin foam model is defined combinatorially in terms of the triangulation of a given manifold or in terms of abstract two-complexes and makes use of the irreducible representations and invariant tensors of a symmetry group. The most carefully studied model in four dimensions in this context is the Barrett–Crane model [1,2] whose Riemannian, *i.e.* $SO(4)$-symmetric, version we consider here. For review articles on the subject, see, for example [3,4].

The spin foam approach was originally developed for pure gravity, in absence of any gauge or matter field, *i.e.* as a description of pure quantum geometry. It is, of course, essential to understand how to couple matter and gauge fields to these models of pure gravity. Ultimately, one wishes to describe the Standard Model matter and interactions in the metric background provided by quantum gravity if a suitable classical limit of the gravity sector is taken. Moreover, the coupling of spin foam gravity to matter might be essential in order to understand various subtle and yet unsolved questions in the area of quantum gravity, and it may even be ultimately required in order to understand the classical limit and therefore to decide which one of several conceivable spin foam models of gravity, all having the same local symmetries, is the correct choice. Finally, such a coupling may lead to new ways of approaching some fundamental and yet unresolved issues of standard particle physics, for example, the hierarchy problem, the cosmological constant problem or a deeper understanding of renormalization.

In this article, we present a spin foam model that couples pure lattice Yang–Mills theory to the Riemannian Barrett–Crane model.

Our strategy is as follows. We start from a lattice formulation of pure Yang–Mills theory whose lattice is given merely combinatorially by the triangulation of a given four-manifold. We do not assume any particular symmetries nor any background metric for this lattice, but rather employ a spin foam model for quantum gravity, here the Barrett–Crane model, in order to describe dynamically the geometry of the triangulation.

All the geometric data necessary for the formulation of Yang–Mills theory on the triangulation are taken from the configuration data of the gravity spin foam model. The coupling of gravity to matter is realized, similarly to the situation at the level of the the classical actions, by writing down Yang–Mills theory for a generic geometry which is given by the gravity sector, while the description of the gravity sector itself is not directly affected by the presence of the gauge fields.

The resulting model of pure gauge theory coupled to quantum gravity retains all the key properties of the gravity model: It is formulated non-perturbatively and relies only on the combinatorics of the triangulation, but does not explicitly refer to any background metric.

The present article is organized as follows. In Section 2, we review general ideas on the coupling of matter to spin foam gravity and place our approach in a wider context. In Section 3, we present our model. We discuss its interpretation and several issues on which it offers a new perspective in Section 4 and conclude in Section 5.

2 Motivation

2.1 Strategies for matter-gravity coupling

At present, the coupling of matter to spin foam gravity is at an exploratory stage. There have been several ideas and proposals, for example:

- the idea that the full unified theory of gravity and matter is a topological quantum field theory, therefore manifestly background independent and in its discrete version triangulation independent [5]. The model of pure gravity would then appear as a sum over only some configurations of the path integral of the unified theory. Indeed, a particular version of the Riemannian Barrett–Crane model is a partial sum over the 'configurations' of the $Spin(4)$-Crane–Yetter invariant [6]. Of course, a realistic unified model should have sufficiently many symmetries in order to incorporate at least the Standard Model of particle physics.
- the idea that matter arises from simplicial geometries, described by spin foams based on abstract two-complexes which do not correspond to smooth manifolds, but only to manifolds with conical singularities [7]. These singularities would then represent particles. The unified theory would therefore be described in terms of spin foams based on abstract two-complexes, for example as given by the formulation of the spin foam model as a Boulatov–Ooguri field theory on a group [8]. Matter would no longer be a separate concept that exists in addition to space-time geometry, but it would rather appear as the structure of singularities in a generalized geometry.
- a proposal for the coupling of various representations of the frame group $SO(4)$ or the spin group $Spin(4)$ to the Barrett–Crane model, again in the picture of a field theory on a group [9]. States of the theory would be given by open spin networks with matter representations attached to their end points, similarly to what has been proposed in the context of loop quantum gravity. Configurations of the path integral, *i.e.* histories, would then include Feynman graphs describing the propagation of these matter representations in addition to the spin foams that are present in the description of gravity.

The latter approach has the advantage that the degrees of freedom that appear in addition to the gravity ones are particular well-specified representations of the frame group which immediately suggests their interpretation as particles of a given spin. However, one has then to explain why, say, spin one particles appear as gauge bosons, and whether these particles have, at least in some limit, the dynamics given by ordinary Yang–Mills theory. One of the problems here is that the concept of a gauge boson as a particle is ultimately a perturbative concept and that we should be able to explain how the Hilbert space of our non-perturbative model can be approximated by a perturbative Fock space. Similar problems arise for spin-$1/2$ representations whose quantum states, at least in a regime in which the gravity sector yields flat Minkowski space and in which the Standard Model sector is perturbative, should admit a Fock space representation and exhibit Fermi–Dirac statistics.

One might hope that there exists enough experience with Lattice Gauge Theory (see, for example [10, 11]) in order to clarify these issues. Unfortunately, one usually heavily relies on fixed hypercubic lattices which represent space-time and which do contain information about a flat background metric. The construction of the weak field or naïve continuum limit which makes contact with the perturbative continuum formulation, relies on the special geometry of

the lattice. The variables of the path integral in lattice gauge theory are the parallel transports $U_\ell = \mathrm{P}\exp(i\int_\ell A_\mu dx^\mu)$ along the links (edges) ℓ of the lattice. In calculating the weak-field limit of lattice gauge theory [10,11], the four components of the vector potential A_μ correspond to the four orthogonal edges attached to each lattice point on the hypercubic lattice. Even though the parallel transport is independent of the background metric, the transition to the perturbative Fock space picture does depend on it. For fermions, the situation is even less transparent, and one faces problems similar to the notoriously difficult question of how to put fermions on the lattice. Whereas in the usual Fock space picture in continuous space-time, the spin statistics relation appears as a consistency condition without any transparent geometric justification, a unified approach to gravity plus matter should provide a construction from which this relation arises naturally, at least in a suitable perturbative limit. These are deep and as yet unresolved questions.

2.2 Our approach

In the view of these conceptual and practical difficulties, we present an alternative and essentially complementary construction for the coupling of 'matter' to spin foam gravity. We concentrate on gauge fields rather than fermions or scalars, *i.e.* on pure Yang–Mills theory. For pure gauge fields, we can circumvent some of the conceptual problems if we focus on the effective behaviour of gauge theory. We rely on the weak field limit of lattice gauge theory and make sure that the gauge theory sector approaches the right continuum limit in an effective sense when the lattice is very fine compared with the gauge theory scale. The model is therefore phenomenologically realistic if the gravity scale is much smaller than the gauge theory scale.

We realize the coupling of pure lattice Yang–Mills theory to the Barrett–Crane model of quantum gravity in the following way. In order to find the relevant geometric data for Yang–Mills theory, we analyze the continuum classical action for the gauge fields and discretize it on a generic triangulation. The required geometric data are then taken, configuration by configuration, from the Barrett–Crane model.

As an illustration, consider a situation in which quantum gravity has a classical limit given by a smooth manifold with Riemannian metric, and assume that we study lattice Yang–Mills theory on this classical manifold, using triangulations that are *a priori* unrelated to the gravity model. Then we require that the continuum limit of this lattice gauge theory agrees with continuum Yang–Mills theory on the manifold that represents the classical limit of gravity.

In order to take the continuum limit including a non-perturbative renormalization of the theory, one sends the bare coupling of Yang-Mills theory to zero and at the same time refines the lattice in a particular way [10, 11]. However, we do not actually pass to the limit, but rather stop when the lattice gets as fine as the triangulation on which the Barrett–Crane model is defined. We assume that we have chosen a very fine triangulation for the Barrett–Crane model and that the Barrett–Crane model assigns geometric data to it that are consistent with the classical limit. This means that its path integral is dominated by configurations whose discrete geometry is well approximated by the Riemannian metric of the smooth manifold in which the triangulation is embedded and which represents the classical limit.

Therefore the geometries that the dominant configurations of the Barrett–Crane model assign to the triangulation, should correspond to the geometry of the triangulation of Yang–Mills theory if we approach the continuum limit of Yang–Mills theory by refining the lattice for Yang–Mills theory more and more. Our strategy is now to *define* Yang–Mills theory on the

same very fine triangulation as the Barrett–Crane model and, configuration by configuration, to use the geometric data from the Barrett–Crane model in the discretized Yang–Mills action.

To be specific, for pure $SU(3)$ Yang–Mills theory interpreted as the gauge fields of QCD, the typical scale is 10^{-13}cm. If the fundamental triangulation is assigned geometric data at the order of the Planck scale, the embedding into the classical limit manifold provides the edges of the triangulation with metric curve lengths of the order of 10^{-33}cm. From the point of view of QCD, this is essentially a continuum limit. In our case, however, the lattice is not merely a tool in order to define continuum Yang–Mills theory non-perturbatively, but we rather have a model with a very fine triangulation that is physically fundamental. This model can be approximated at large distances by a smooth manifold with metric and continuous Yang–Mills fields on it.

3 The model in detail

3.1 The Barrett–Crane model

Let us recall the basic ideas of the Barrett-Crane model for quantum gravity [1, 2]. In the simplest form, the Barrett–Crane model is formulated for a given combinatorial triangulation of a four-manifold or alternatively for the two-complex dual to it. The geometry of the triangulation is made dynamical. Therefore the model specifies a path integral whose configurations are geometries that can be assigned to the given triangulation. It can be understood [1,2,3,12,13,14,15,16] as a quantum analogue of the formulation of classical gravity as an $SO(4)$ BF-theory with constraints [17,12,18] in a discrete setting.

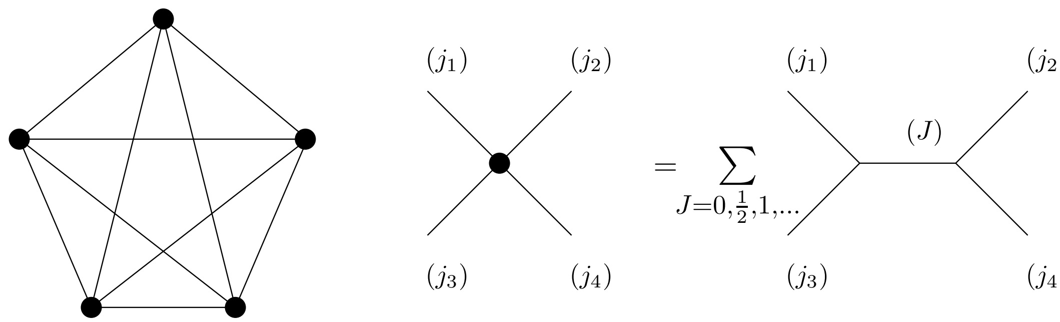
The starting point is the quantization of $SO(4)$ BF-theory on the triangulation of a four-manifold. The fields of BF-theory are an $SO(4)$-gauge connection and the B-field, an $\mathfrak{so}(4)$-valued two-form B. In the simplicial setting, this two-form is represented by an assignment of a value $B(t) = B^{IJ}(t)\,T_{IJ} \in \mathfrak{so}(4)$ for each triangle t,

$$B(t) = \int\limits_t B. \tag{3.1}$$

Above we have chosen a basis of antisymmetric real 4×4 matrices $\{T_{IJ}\}_{IJ}$, $0 \leq I < J \leq 3$, for $\mathfrak{so}(4)$.

The construction of a discretized version of classical gravity from this BF-theory consists of four steps. Firstly, the gauge group $SO(4)$ is supposed to coincide with the frame group. The B-field which up to this point lived merely in some internal space $\mathfrak{so}(4)$, now represents bi-vectors $\Lambda^2(\mathbb{R}^4)$ constructed from tangent vectors in $\mathbb{R}^4$. This step involves the identification $\Lambda^2(\mathbb{R}^4) \cong \mathfrak{so}(4)^*$, $v_I \wedge v_J \leftrightarrow T^{IJ}$ where $\{v_I\}_I$ forms an orthonormal basis of $\mathbb{R}^4$ and $\{T^{IJ}\}_{IJ}$ is a basis of $\mathfrak{so}(4)^*$ dual to $\{T_{IJ}\}_{IJ}$.

Secondly, one has to implement the gravity constraints. These are conditions on the $B(t)$ which translate in the above geometrical picture to the natural consistency conditions that the triangles described by the bi-vectors $B(t) \in \Lambda^2(\mathbb{R}^4)$ actually form the tetrahedra and four-simplices of a triangulated manifold. The $\mathfrak{so}(4)$-valued $B(t)$ assigned to the triangles t then represent bi-vectors $B(t) = *(u \wedge w) = B^{IJ}(t) v_I \wedge v_J \in \Lambda^2(\mathbb{R}^4)$ which describe the position of the triangle up to translation in $\mathbb{R}^4$. This means that the triangle is spanned by the vectors $u, w \in \mathbb{R}^4$ and has the area $\frac{1}{2}||B(t)||$. The coefficients $B^{IJ}(t)$ can be understood



(a)

Figure 1: (a) The $10j$-symbol of the Barrett–Crane model. (b) The four-valent Barrett–Crane intertwiner for balanced representations $(j) = V_j \otimes V_j$ of $SO(4)$ and its tree decomposition. The three-valent vertices without dot indicate the Clebsh–Gordan coupling of $SO(4)$-representations.

as the discrete counterpart of the wedge product of co-tetrad fields, $B^{IJ} = *(e^I \wedge e^J)$, defined on the triangle. The star $*$ denotes the Hodge operator $*(e^I \wedge e^J) = \frac{1}{2}\varepsilon^{IJ}{}_{KL} e^K \wedge e^L$.

Thirdly, one wishes to study a quantum theory of the geometry described above which is given in the language of a path integral. Consider the spin foam model of $SO(4)$ BF-theory on the two-complex dual to the given triangulation. It provides a path integral which is the sum over all configurations. The configurations are all possible assignments of finite-dimensional irreducible representations of $SO(4)$ to the triangles and all possible assignments of compatible intertwiners (invariant tensors) of $SO(4)$ to the tetrahedra. The finite-dimensional irreducible representations of $SO(4)$ can be written as $V_j \otimes V_{j'}$ where j, j' are half-integers, $j + j'$ is integer, and $V_j \cong \mathbb{C}^{2j+1}$ denote the finite-dimensional irreducible representations of $SU(2)$. Under the path integral, there are weights, often called *amplitudes*, for each triangle, tetrahedron and four-simplex. The basis vectors T^{IJ} of $\mathfrak{so}(4)^*$ of the classical theory correspond in this path integral picture to the generators $\widehat{T}^{IJ}$ of $\mathfrak{so}(4)$ acting on the representation assigned to the relevant triangle.

The fourth step is the implementation of the gravity constraints at the quantum level [1,2]. The constraints restrict the sum over representations to the *balanced* (also called *simple*) irreducible representations of $SO(4)$. These are the representations that are of the form $V_j \otimes V_j$. The intertwiners are restricted to the so-called *Barrett–Crane intertwiner* (Figure 1). The partition function of the Barrett–Crane model for a given triangulation reads,

$$Z = \Bigl(\prod_t \sum_{j_t=0,\frac{1}{2},1,\ldots}\Bigr) \Bigl(\prod_t \mathcal{A}_t^{(2)}(\{j_t\})\Bigr) \Bigl(\prod_\tau \mathcal{A}_\tau^{(3)}(\{j_t\})\Bigr) \Bigl(\prod_\sigma \mathcal{A}_\sigma^{(4)}(\{j_t\})\Bigr). \tag{3.2}$$

Here the products are over all triangles t, tetrahedra τ and four-simplices σ of the triangulation. As opposed to the spin foam model of BF-theory whose amplitudes are uniquely determined, here the geometrical constraints do not completely fix the amplitudes. Therefore under the path integral, we write generic amplitudes $\mathcal{A}_t^{(2)}$, $\mathcal{A}_\tau^{(3)}$, $\mathcal{A}_\sigma^{(4)}$ for each triangle, tetra-

hedron and four-simplex which can depend on the representations $V_{j_t}\otimes V_{j_t}$ that are associated to the triangles t.

The triangle amplitude is normally chosen to be $\mathcal{A}_t^{(2)} = (2j_t+1)^2 = \dim V_{j_t}\otimes V_{j_t}$ as in BF-theory. The four-simplex amplitude is given by a special $10j$-symbol of balanced $SO(4)$-representations (Figure 1) which is formed from Barrett–Crane intertwiners. We consider here the second version of the model presented in [1] in which the two Barrett–Crane intertwiners associated with each tetrahedron, one in either of the two attached four-simplices, are independent.

Several variations of the model have been studied [1,8,19] which differ in their tetrahedron amplitudes, obtained either from ideas of lattice gauge theory [13, 14, 15] or from particularly simple actions in the formulation as a field theory on a group [8, 19] (note that papers written in the language of a field theory on a group use the two-complex dual to the triangulation).

The notation used in (3.2) is still somewhat ambiguous as it does not specify all relative orientations that are necessary for a consistent definition. A formula which shows all details explicitly was given, for example, in [15].

The configurations of the path integral (3.2) are interpreted as the histories of the gravitational field. The balanced representations $V_j\otimes V_j$, $j=0,\frac{1}{2},1,\ldots$, which are assigned to the triangles, describe the areas $\ell_P^2\sqrt{j(j+1)}$ of the triangles. Here ℓ_P is a length at the order of the Planck length. The expression $\sqrt{j(j+1)}$ for the area is a consequence of the quantization procedure of [20] and agrees with the results of loop quantum gravity. Recently, it was proposed to use $j+\frac{1}{2}$ instead [21] which has the same large-j asymptotics and coincides with the expressions for the area used in the Regge action.

While the areas of the triangles are the fundamental geometric quantities of the Barrett–Crane model, one can also extract other metric data of the manifold (holonomies, three-volume, *etc.*) from the path integral (3.2) or from its dual connection formulation [15].

3.2 Discretized pure Yang–Mills theory

Let us now consider the classical continuum Yang–Mills action for pure gauge fields on a Riemannian four-manifold M. The gauge group is denoted by G and its Lie algebra by $\mathfrak{g}$. We use the Euclidean (imaginary) time formulation,

$$S = \frac{1}{4g_0^2}\int\limits_M \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\sqrt{\det g}\,d^4x = \frac{1}{4g_0^2}\int\limits_M \mathrm{tr}(F\wedge *F). \tag{3.3}$$

Here we write $F=F_{\mu\nu}\,dx^\mu\wedge dx^\nu$, $\mu,\nu=0,\ldots,3$, for the field strength two-form using any coordinate basis $\{dx^\mu\}_\mu$. The action makes use of metric data as it involves the Hodge star operation.

We reformulate this action in order to arrive at a path integral quantum theory which can be coupled to the Barrett–Crane model. This is done in two steps.

Firstly, we consider the preliminary step towards the Barrett–Crane model, outlined in Section 3.1, in which classical variables $B(t)=B^{IJ}(t)T_{IJ}\in\mathfrak{so}(4)$ are attached to the triangles which are interpreted as the bi-vectors $B^{IJ}(t)v_I\wedge v_J\in\Lambda^2(\mathbb{R}^4)$ that span the triangles in $\mathbb{R}^4$.

Therefore we discretize the Yang–Mills action (3.3) on a generic combinatorial triangulation. We mention that for generic triangulations with respect to a flat background metric, there exists a formalism in the context of gauge theory on random lattices [22]. Here we need a formulation which does not refer to any particular background metric.

We pass to locally orthonormal coordinates, given by the co-tetrad one-forms $\{e^I\}_I$, $I = 0,\ldots,3$, *i.e.* $dx^\mu = c^\mu_I e^I$, and obtain

$$
\begin{aligned}
S &= \frac{1}{8g_0^2}\int\limits_M \mathrm{tr}(F_{IJ}F_{KL})\varepsilon^{KL}{}_{MN}e^I\wedge e^J\wedge e^M\wedge e^N \\
&= \frac{1}{4g_0^2}\int\limits_M \sum_{I,J,M,N}\mathrm{tr}(F_{IJ}^2)\varepsilon_{IJMN}*(e^I\wedge e^J)\wedge*(e^M\wedge e^N),
\end{aligned}
\tag{3.4}
$$

where $F_{IJ} = F_{\mu\nu}c^\mu_I c^\nu_J\, e^I\wedge e^J$. In the last step, we have made use of the symmetries of the wedge product and of the $*$-operation, and there are no summations other than those explicitly indicated, in particular there is no second sum over I,J.

Discretization of (3.4) turns integration over M into a sum over all four-simplices,

$$
S = \sum_\sigma S_\sigma. \tag{3.5}
$$

Two-forms with values in $\mathfrak{g}$ and $\mathfrak{so}(4)$ correspond to a colouring of all triangles t with values $F(t)\in\mathfrak{g}$ and $B(t)\in\mathfrak{so}(4)$, respectively.

Equation (3.4) resembles a preliminary step in the construction of the four-volume operator in [23]. The total volume of M is given by

$$
V = \int\limits_M \sqrt{\det g}\,dx^1\wedge dx^2\wedge dx^3\wedge dx^4 = \frac{1}{4!}\int\limits_M \varepsilon_{IJMN}*(e^I\wedge e^J)\wedge*(e^M\wedge e^N). \tag{3.6}
$$

Discretization results in

$$
V = \sum_\sigma \frac{1}{30}\sum_{t,t'}\frac{1}{4!}\varepsilon_{IJMN}\,\mathrm{sgn}(t,t')T^{IJ}(t)T^{MN}(t'), \tag{3.7}
$$

where the sums are over all four-simplices σ and over all pairs of triangles (t,t') in σ that do not share a common edge. The wedge product of co-tetrad fields $*(e^I\wedge e^J)$ was replaced by a basis vector T^{IJ} of the $\mathfrak{so}(4)^*$ that is associated to the given triangle t, and the wedge product of two of them is implemented by considering pairs (t,t') of complementary triangles with a sign factor $\mathrm{sgn}(t,t')$ depending on their combinatorial orientations. Let (12345) denote the oriented combinatorial four-simplex σ and $(PQRST)$ be a permutation π of (12345) so that $t=(PQR)$ and $t'=(PST)$ (two triangles t,t' in σ that do not share a common edge have one and only one vertex in common). Then the sign factor is defined by $\mathrm{sgn}(t,t')=\mathrm{sgn}\,\pi$ [23].

The boundary of a given four-simplex σ is a particular three-manifold and can be assigned a Hilbert space [20] which is essentially a direct sum over all colourings of the triangles t of σ with balanced representations $V_{j_t}\otimes V_{j_t}$, $j_t = 0,\frac{1}{2},1,\ldots$ of $SO(4)$. From (3.7), one obtains a four-volume operator,

$$
\widehat{V}_\sigma = \frac{1}{30}\sum_{t,t'}\frac{1}{4!}\varepsilon_{IJMN}\,\mathrm{sgn}(t,t')\widehat{T}^{IJ}(t)\widehat{T}^{MN}(t'), \tag{3.8}
$$

where the $\mathfrak{so}(4)$-generators $\widehat{T}^{IJ}$ act on the representation $V_{j_t}\otimes V_{j_t}$ associated to the triangle t. $\widehat{V}_\sigma$ is an operator on the vector space

$$
\mathcal{H}_\sigma = \bigotimes_t V_{j_t}\otimes V_{j_t}, \tag{3.9}
$$

of one balanced representation $V_{j_t}\otimes V_{j_t}$ for each triangle t in σ. The space $\mathcal{H}_\sigma$ is an intermediate step in the implementation of the constraints [20] where only the simplicity condition has been taken into account.

Observe that the sum over all pairs of triangles (t,t') provides us with a particular symmetrization which can be thought of as an averaging over the angles[1] that would be involved in an exact calculation of the volume of a four-simplex.

3.3 The coupled model

We are interested in a discretization of the Yang–Mills action (3.4) which can be used in a path integral quantization, *i.e.* we wish to obtain a number (the value of the action) for each combined configuration of gauge theory and the Barrett–Crane model. In the case of the four-volume, equation (3.8) provides us with an operator for each four-simplex σ. The analogous operator obtained from (3.4) reads,

$$\widehat{S}_\sigma = \frac{1}{4g_0^2}\,\frac{1}{30}\sum_{t,t'} \mathrm{tr}(F(t)^2)\varepsilon_{IJMN}\,\mathrm{sgn}(t,t')\widehat{T}^{IJ}(t)\widehat{T}^{MN}(t'). \tag{3.11}$$

Since in (3.4) only the field strength components F_{IJ}, but not F_{MN} appear, we need $F(t)$ only for one of the two triangles.

How to extract one number for each configuration from it? $\widehat{S}_\sigma$ is not merely a multiple of the identity operator so that it does not just provide a number for each assignment of balanced representations to the triangles. It is therefore useful to generalize the sum over configurations of the path integral so that it not only comprises a sum over irreducible representations attached to the triangles, but also a sum over a basis for each given representation. This second sum is just the trace of $\widehat{S}_\sigma$ over the vector space $\mathcal{H}_\sigma$. The value of the action to use in the path integral is therefore,

$$S_\sigma = \frac{1}{\dim\mathcal{H}_\sigma}\,\mathrm{tr}_{\mathcal{H}_\sigma}(\widehat{S}_\sigma). \tag{3.12}$$

In a lattice path integral, the total Boltzmann weight for the Yang–Mills sector is therefore the product,

$$\exp\bigl(-\sum_\sigma S_\sigma\bigr) = \prod_\sigma \exp(-S_\sigma), \tag{3.13}$$

over all four-simplices. An alternative prescription to (3.12) and (3.13) is given by,

$$\prod_\sigma \frac{1}{\dim\mathcal{H}_\sigma}\,\mathrm{tr}_{\mathcal{H}_\sigma}\exp(-\widehat{S}_\sigma). \tag{3.14}$$

[1] An alternative expression for the four-volume from the context of a first order formulation of Regge calculus [24] is given by,

$$(V_\sigma)^3 = \frac{1}{4!}\varepsilon^{abcd} b_a\wedge b_b\wedge b_c\wedge b_d,. \tag{3.10}$$

where the indices a,b,c,d run over four out of the five tetrahedra of the four-simplex σ (the result is independent of the tetrahedron which is left out), and the b_a are vectors normal to the hyperplanes spanned by the tetrahedra whose lengths are proportional to the three-volumes of the tetrahedra. This formulation favours the angles between the b_a over the quantized areas and fits into the dual or connection formulation of the Barrett–Crane model.

Whereas (3.13) provides the average of the eigenvalues of the operator $\widehat{S}_\sigma$ in the exponent, the trace in (3.14) can be understood as a sum over different configurations each contributing a Boltzmann weight $\exp(-S_\sigma)$ with a different eigenvalue of the four-volume. We stick to (3.13) as this expression is closest to the classical action.

We note that the operator $\widehat{S}_\sigma$ of (3.11) is Hermitean, diagonalizable and $\mathfrak{so}(4)$-invariant. This can be seen for each of its summands if one applies the splitting $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ of $\mathfrak{so}(4)$ into a self-dual and an anti-self dual part. Then

$$\varepsilon_{IJMN}\widehat{T}^{IJ} \otimes \widehat{T}^{MN} = 4\sum_{k=1}^{3}(\widehat{J}_k^+ \otimes \widehat{J}_k^+ + \widehat{J}_k^- \otimes \widehat{J}_k^-). \tag{3.15}$$

Here $J_k^\pm$, $k = 1, 2, 3$, denote the generators of the (anti)self-dual $\mathfrak{su}(2)$. Invariance under $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ follows from the fact that for a tensor product $V_j \otimes V_\ell$ of irreducible $\mathfrak{su}(2)$-representations,

$$\sum_k \widehat{J}_k \otimes \widehat{J}_k = \frac{1}{2}\big(j(j+1) + \ell(\ell+1) - C^{(2)}_{V_j\otimes V_\ell}\big), \tag{3.16}$$

where $C^{(2)}_{V_j\otimes V_\ell}$ denotes the quadratic Casimir operator of $\mathfrak{su}(2)$ on $V_j \otimes V_\ell$. This argument holds independently for the self-dual and anti self-dual tensor factors.

There is a further possible choice for an extraction of the four-simplex volume from the Barrett–Crane configurations (We thank A. Mikovic for a request to clarify this construction). We could insert the operator into the $10j$-symbol itself, *i.e.* in the Barrett–Crane vertex amplitude. This means contracting 20 indices,

$$\prod_\sigma B_{i_1i_2i_3i_4}B_{j_4i_5i_6i_7}B_{j_7j_3i_8i_9}B_{j_9j_6j_2i_{10}}B_{j_{10}j_8j_5j_1}[e^{-\widehat{S}}]_{i_1j_1i_2j_2\ldots i_{10}j_{10}}. \tag{3.17}$$

This way of coupling Yang–Mills theory to the Barrett–Crane model is more natural from the spin foam point of view since it just amounts to a spin network evaluation. In fact it was shown in [32] to be derivable directly from a path integral using the formalism of [33].

So far, we have prescribed how the Yang–Mills path integral obtains its geometric information from the Barrett–Crane model which is required to formulate the discretization of the action (3.4). The curvature term $\mathrm{tr}(F(t)^2)$ of the Yang–Mills connection in (3.11) can be treated as usual in Lattice Gauge Theory with Wilson action.

Associate elements of the gauge group $g_e \in G$ to the edges e of the triangulation which represent the parallel transports of the gauge connection. Calculate the holonomies $g(t)$ around each triangle t for some given orientation. Then the curvature term arises at second order in the expansion of the holonomy [10, 11],

$$\mathrm{Re}\,\mathrm{tr}\,g(t) \sim \mathrm{Re}\,\mathrm{tr}\left(\mathbb{1} + ia_tF(t) - \frac{a_t^2}{2}F(t)^2 + \cdots\right) = d - \frac{a_t^2}{2}\,\mathrm{tr}\,F(t)^2 + \cdots, \tag{3.18}$$

where a_t denotes the area of the triangle t. Here the tr is evaluated in a representation of dimension d of G.

The area a_t of a triangle t is easily obtained from the data of the Barrett–Crane model by $a_t^2 = j_t(j_t+1)$, ignoring all prefactors (or by the alternative choice $a_t^2 = (j_t + \frac{1}{2})^2$).

For each four-simplex, we therefore obtain the Yang–Mills amplitude (or Boltzmann weight),

$$\mathcal{A}_\sigma^{(YM)} = \exp\left(\beta\sum_{t,t'}\frac{\mathrm{Re}\,\mathrm{tr}\,g(t) - d}{j_t(j_t+1)}\varepsilon_{IJMN}\frac{1}{\dim\mathcal{H}_\sigma}\,\mathrm{tr}_{\mathcal{H}_\sigma}\big(\widehat{T}^{IJ}(t)\widehat{T}^{MN}(t')\big)\right), \tag{3.19}$$

where β is a coupling constant which absorbs all prefactors and the bare gauge coupling constant. The fundamental area scale, ℓ_P^2, cancels because we have divided a four-volume by a square of areas. Observe that the geometric coupling in the exponent, a volume divided by a square of an area, is essentially the same as in random lattice gauge theory [22].

Note two special cases. Firstly, for a flat gauge connection we have $g(t) = \mathbb{1}$ so that the Boltzmann weight is trivial, $\mathcal{A}_\sigma^{(YM)} = 1$. In this case, we recover the Barrett–Crane model without any additional fields. Secondly, if a given configuration of the Barrett–Crane model corresponds to a flat metric and the triangulation is chosen to be regular, for example obtained by subdividing a hypercubic lattice, then the four-volume is essentially the area squared of a typical triangle,

$$\sum_{t,t'} \varepsilon_{IJMN} \frac{1}{\dim \mathcal{H}_\sigma} \operatorname{tr}_{\mathcal{H}_\sigma} \big(\widehat{T}^{IJ}(t)\widehat{T}^{MN}(t')\big) \sim j_t(j_t+1)\cdot \text{const.} \tag{3.20}$$

In this case, the Yang–Mills amplitude reduces to the standard Boltzmann weight of lattice gauge theory,

$$\mathcal{A}_\sigma^{(YM)} = \exp\Big(\beta' \sum_t (\operatorname{Re}\operatorname{tr} g(t) - d)\Big). \tag{3.21}$$

The model of Yang–Mills theory coupled to the Barrett–Crane model is finally given by the partition function

$$Z = \Big(\prod_e \int_G dg_e\Big)\Big(\prod_t \sum_{j_t=0,\frac{1}{2},1,\ldots}\Big)\Big(\prod_t \mathcal{A}_t^{(2)}\Big)\Big(\prod_\tau \mathcal{A}_\tau^{(3)}\Big)\Big(\prod_\sigma (\mathcal{A}_\sigma^{(4)}\,\mathcal{A}_\sigma^{(YM)})\Big). \tag{3.22}$$

In addition to the Barrett–Crane model of pure gravity, we now have the path integral of lattice gauge theory, one integration over G for each edge e, and the Boltzmann weight $\mathcal{A}_\sigma^{(YM)}$ of Yang–Mills theory with one factor for each four-simplex in the integrand.

The observables of the gauge theory sector of the coupled model are, as usual, expectation values of spin network functions under the path integral (3.22).

3.4 The coupled model as a spin foam model

While the model (3.22) is a hybrid involving a lattice gauge theory together with a spin foam model of gravity, we can make use of the strong-weak duality transformation of lattice gauge theory [25, 26, 27] in order to obtain a single spin foam model with two types of 'fields'.

Therefore we split the gauge theory amplitudes so that

$$\prod_\sigma \mathcal{A}_\sigma^{(YM)} = \prod_t \mathcal{A}_t^{(YM)}, \tag{3.23}$$

where the second product is over all triangles and

$$\mathcal{A}_t^{(YM)} := \exp\Big(\beta n_t \frac{\operatorname{Re}\operatorname{tr} g(t) - d}{j_t(j_t+1)} \sum_{t'} \varepsilon_{IJMN} \frac{1}{\dim \mathcal{H}_\sigma} \operatorname{tr}_{\mathcal{H}_\sigma}(\widehat{T}^{IJ}(t)\widehat{T}^{MN}(t'))\Big). \tag{3.24}$$

Here n_t denotes the number of four-simplices that contain the triangle t, and the sum is over all triangles t' that are contained in the same four-simplex as t, but do not share an edge with t.

We can apply the duality transformation to the gauge theory sector of the coupled model (3.22) and obtain,

$$
\begin{aligned}
Z &= \Big(\prod_t \sum_{\rho_t}\Big)\Big(\prod_e \sum_{I_e}\Big)\Big(\prod_t \sum_{j_t=0,\frac{1}{2},1,\dots}\Big)\Big(\prod_t (\mathcal{A}_t^{(2)}\widehat{\mathcal{A}}_t^{(YM)})\Big)\\
&\quad\times\Big(\prod_\tau \mathcal{A}_\tau^{(3)}\Big)\Big(\prod_\sigma \mathcal{A}_\sigma^{(4)}\Big)\Big(\prod_v \mathcal{A}_v^{(YM)}(\{\rho_t, I_e\})\Big).
\end{aligned}
\tag{3.25}
$$

Here $\widehat{\mathcal{A}}_t^{(YM)}$ are the character expansion coefficients of $\mathcal{A}_t^{(YM)}$ as functions of $g(t)$. For example, for $G=U(1)$, we have

$$
\widehat{\mathcal{A}}_t^{(YM)} = I_{k_t}(\gamma)e^{-\gamma d},\qquad \gamma=\frac{\beta\, n_t}{j_t(j_t+1)}\sum_{t'}\varepsilon_{IJMN}\frac{1}{\dim\mathcal{H}_\sigma}\,\mathrm{tr}_{\mathcal{H}_\sigma}(\widehat{T}^{IJ}(t)\widehat{T}^{MN}(t')), \tag{3.26}
$$

where I_k denote modified Bessel functions and the irreducible representations are characterized by integers $k_t\in\mathbb{Z}$ for each triangle t. Similarly for $G=SU(2)$,

$$
\widehat{\mathcal{A}}_t^{(YM)} = 2(2\ell_t+1)I_{2\ell_t+1}(\gamma)e^{-\gamma d}/\gamma, \tag{3.27}
$$

where $\ell_t=0,\frac{1}{2},1,\dots$ characterize the irreducible representations of $SU(2)$. Note that these coefficients depend via γ on the assignment of balanced representations $\{j_t\}$ to the triangles. The path integral now consists of a sum over all colourings of the triangles t with irreducible representations of the gauge group G and over all colourings of the edges e with compatible intertwiners of G as well as of a sum over all colourings of the triangles with balanced representations of $SO(4)$. Under the path integral, there are in addition amplitudes $\mathcal{A}_v^{(YM)}$ for each vertex which can be calculated from the representations and intertwiners at the triangles and edges attached to v. The $\mathcal{A}_v^{(YM)}$ are very similar to the four-simplex amplitudes of Figure 1(a), just using the intertwiners attached to the edges incident in v. For more details, see [25, 27] where the $\mathcal{A}_v^{(YM)}$ are called $C(v)$. The observables of lattice gauge theory can be evaluated as indicated in [25, 27].

Observe that in (3.25), simplices at several levels are coloured, namely triangles with irreducible representations of the gauge group G and with balanced representations of $SO(4)$, edges with compatible intertwiners of G and tetrahedra with Barrett–Crane intertwiners (hidden in the $\mathcal{A}_\sigma^{(4)}$). The model (3.25) therefore does not admit a formulation involving only two-complexes. The technology of the field theory on a group formulation would have to be significantly extended, namely at least to generate three-complexes, before it can be applied to the model (3.25). Observe furthermore that we now have amplitudes at all levels from vertices v to four-simplices σ.

4 Discussion

4.1 Features of the model

We now discuss briefly the main features of the coupled model. Firstly, it shares the main characteristics of spin foam models for pure gravity: it is formulated without reference to any background metric, using only the combinatorial structure of a given triangulation of a four-manifold as well as algebraic data from the representation theory of the frame group of gravity,

here $SO(4)$, and of the gauge group G of Yang–Mills theory. The partition function (3.25) is well defined on any finite triangulation and formulated in non-perturbative terms.

The general discretization procedure we have used in order to write down lattice gauge theory in the geometry specified by the spin foam model of gravity, is also applicable to other spin foam models of geometry and, moreover, to theories other than pure gauge theory as long as they can be reliably studied in a discrete setting.

As we have explained, we use a probability interpretation of the partition function similar to Statistical Mechanics. For lattice gauge theory, this is the natural thing to do, and for the Barrett–Crane model it is, at least technically, justified by the positivity result of [28] and very similar to what has been suggested in the three-dimensional case [29].

The choice of the Riemannian gravity model just forces us to use a formulation of lattice gauge theory in a Riemannian signature. This coincides with what is usually done in the imaginary time picture, but it is not restricted to that case. The use of amplitudes e^{iS} rather than probabilities e^{-S} in the Yang–Mills sector would be perfectly possible and correspond to the Feynman path integral of quantum Yang–Mills theory on a Riemannian manifold which is a toy model just as Riemannian gravity is one. Replacing the Riemannian Barrett–Crane model by the Lorentzian version in which all triangles are space-like would also force us to use the Feynman path integral for Yang–Mills theory rather than the Statistical Mechanics path integral and in addition change the Yang–Mills action accordingly.

The structure of the coupled model (3.25) reflects the fact that the action of classical gravity coupled to classical Yang–Mills theory is the action of pure gravity plus the action of Yang–Mills theory in curved space-time. Indeed, the amplitudes for the gravity sector are unaffected by the coupling whereas those of the gauge theory sector acquire a dependence on the representations labelling the gravity configurations, *i.e.* they depend on the four-geometries that represent the histories of the gravitational field. Interestingly, the data we need in order to specify this coupling, are only areas of triangles and volumes of four-simplices.

Since the labellings used in the coupled model (3.25) make use of more than two levels of the triangulation, there is no easy way of a "GUT-type" unification of gravity and Yang–Mills theory by just studying a bigger symmetry group which contains both the frame group of gravity and the gauge group of Yang–Mills theory. The problem is here that gauge theory in its connection formulation lives on the edges and triangles of the given triangulation while the $SO(4)$ BF-theory from which the Barrett–Crane model is constructed, naturally lives on the two-complex dual to the triangulation. Gravity and Yang–Mills theory therefore retain separate path integrals and are coupled only by the amplitudes.

Finally, the point of view of effective theories we have chosen in the construction of the coupled model might mean that our strategy is *only* valid at an effective level, but not the final answer microscopically. The model may, however, still form an important intermediate step in the construction of the classical limit and be relevant also to other microscopic approaches of coupling matter to gravity if these models are studied at large distances.

We remark that the model does depend on the chosen triangulation because already the Barrett–Crane model does. A practical solution might be that the long range or low energy effective behaviour turns out not to depend on the details of the triangulation. More strongly, one can pursue approaches such as a refinement and renormalization procedure or a sum over triangulations in order to make the microscopic model independent of the triangulation.

4.2 Spin foam models in the presence of matter

Several aspects of quantum gravity are obviously affected by the presence of matter in the model, changing the answer to several questions from the context of pure gravity. For example, it was studied which is the dominant contribution to the path integral of the Barrett–Crane model. Numerical calculations [30] using the Perez–Rovelli version [19] of the Barrett–Crane model show a dominance of $j_t = 0$ configurations which correspond to degenerate geometries if the $\sqrt{j_t(j_t+1)}$ are interpreted as the areas of the triangles. One might think that this degeneracy can be avoided by just using the alternative interpretation, taking $j_t + \frac{1}{2}$ to be the areas, so that most triangles have areas of Planck size. However, independent of this interpretation, also the reformulation of the Barrett–Crane model in the connection picture [15] indicates problems with geometrically degenerate configurations. This situation may well change if matter is included in the model, and it will also affect the construction of a classical limit. Also the divergence of the partition function of the version of De Pietri *et al.* [12] and the classical limit will be affected by the presence of matter.

4.3 Planck scale versus QCD scale

Just as many questions in quantum gravity are affected by the presence of matter and gauge fields, many issues in gauge theory have to be rethought or rephrased when the coupling with gravity is considered. Here we briefly discuss some of the questions we face if we compare the gauge theory sector of the coupled model (3.25) with a realistic theory and interpret it as the pure gauge fields of QCD.

In the standard formulation of lattice Yang–Mills theory, the (hypercubic) lattice is considered as a purely technical tool in order to define the continuum theory in a non-perturbative way. Starting with some lattice with a spatial cut-off given by the lattice spacing a, one wishes to construct a continuum limit in which the lattice is refined while the relevant physical quantities are kept fixed. These physical quantities are, for example, the masses of particles $m_j = 1/\xi_j a$ which are given by the Euclidean correlation lengths ξ_j which we specify in terms of multiples of the lattice constant. In pure QCD, quantities of this type are the glue balls.

One tunes the bare parameters of the theory towards a critical point, *i.e.* to a value where the relevant correlation lengths ξ_j diverge. This allows a refinement of the lattice, $a \to 0$, while the observable masses m_j are kept constant. Taking this limit removes the cut-off and non-perturbatively renormalizes the theory.

In a model in which lattice Yang–Mills theory is coupled to gravity, we are no longer interested in actually taking this continuum limit. The triangulation is now rather a fundamental structure with a typical length scale of the order of the Planck length, for example obtained by dynamically assigning areas to the triangles as in the Barrett–Crane model. Instead of the continuum limit, we now have to consider a continuum approximation in which the long distance behaviour of Yang–Mills theory (long distances compared with the Planck length) is approximated by a continuum theory, very similar to common situations in condensed matter physics in which there exist underlying crystal lattices.

Coming back to our example in which we interpret the gauge theory sector as QCD, we first have to explain why the ratio $m_{\text{Planck}}/m_{\text{QCD}} \sim 10^{20}$ is so big, where m_{QCD} is a typical mass generated by QCD, or why the typical correlation length of QCD, $\xi_{\text{QCD}} \sim 10^{20}$, is so large in Planck units (see [31] for some not so common thoughts on this issue).

One solution would be to employ a fine-tuning mechanism in the combined full quantum

model. There could be a parameter (maybe not yet discovered in the formulations of the Barrett–Crane model) which has to be fine-tuned to make the combined model almost critical and to achieve exactly the right correlation length ξ_{QCD}. The coupled model (3.25), for example, contains the bare parameter β which enters the $\widehat{\mathcal{A}}_t^{(YM)}$ and which originates from the inverse temperature of lattice Yang–Mills theory. This β is a candidate for such a fine-tuning procedure.

However, there might be a way of avoiding any fine tuning. Looking at the structure of the Yang–Mills amplitude (3.24), one could drop β from that expression and rather consider an effective

$$\beta_{\mathrm{eff}} = \frac{n_t \langle V \rangle}{\langle a^2 \rangle}, \tag{4.1}$$

of Yang–Mills theory which originates from the geometric data of the gravity sector, say, via suitable mean values for four-volume V and area square a^2. From perturbation theory at one loop, the typical correlation length of QCD in lattice units scales with the bare inverse temperature β as

$$\xi_{\mathrm{QCD}} = \xi_0 \exp\big(\frac{8\pi^2}{11}\beta\big), \tag{4.2}$$

where the prefactor ξ_0 depends on the details of the action and of the lattice. A rough estimate shows that one can reach $\xi_{\mathrm{QCD}} \sim 10^{20}$ already with $\beta \sim 10^1$. It is therefore tempting to drop the last coupling constant from our toy model of QCD and to make use of the gravity sector in order to provide an effective coupling constant $\beta_{\mathrm{eff}} \sim 10^1$ for QCD. As suggested in [31], one should reverse the argument and ask what is the effective QCD coupling constant at the Planck scale. In the coupled spin foam model this corresponds to extracting β_{eff} from the small-j regime of the gravity sector. This might be an elegant way of generating a large length scale and an almost critical behaviour without fine-tuning.

The crucial question is whether the long distance behaviour of the coupled model is stable even though the effective coupling constant β_{eff} of the gauge theory sector is affected by quantum fluctuations of the geometry. This is the non-perturbative way of rephrasing the question why the renormalizability of QCD is not spoiled by quantum fluctuations of the geometry at short distances. From random lattice gauge theory on a triangulation with fixed geometry, *i.e.* without quantum fluctuations, we expect that the large distance behaviour at $\beta_{\mathrm{eff}} \sim 10^1$ is described by an almost critical lattice gauge theory and thus by universality arguments largely independent of the microscopic details. If this situation persists as the geometry becomes dynamical, then the gauge theory sector will still be almost critical. In particular, a correlation function over 10^{20} triangles has to be independent of the microscopic quantum fluctuations of the geometry. This is a test of whether the coupled model can solve the hierarchy problem, *i.e.* in our language whether it can generate an exponentially large scale that is stable under microscopic fluctuations of the quantum geometry. The same mechanism would then also predict a dependency of the observed coupling ‘constant’ α_s on the geometry of space-time, *i.e.* potentially explain varying constants in particle physics.

5 Conclusion

We have outlined a procedure for coupling Yang–Mills theory to Riemannian spin foam quantum gravity, motivated from the long distance behaviour of lattice gauge theory, and obtained

a spin foam model (3.25) which implements the coupling of pure gauge theory to the Barrett–Crane model.

The obvious further questions to ask concern the ground state of the coupled model compared to the ground state of the pure gravity model, the existence of propagating modes and the question of the typical correlation lengths of the gauge theory sector. Furthermore, it would be an interesting project to extend the approach to include other matter field such as scalars and fermions and to explore how it relates to other strategies of coupling matter to spin foam models.

A first step towards experimentally relevant applications might be the special case of pure $U(1)$ gauge fields which in one phase of the theory describes free photons. The possibility of detecting quantum gravity effects from modified dispersion relations of various particles, in particular photons, has recently attracted a lot of attention. The coupled model could serve as a first test case in order to study such effects for the case of spin foam quantum gravity.

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